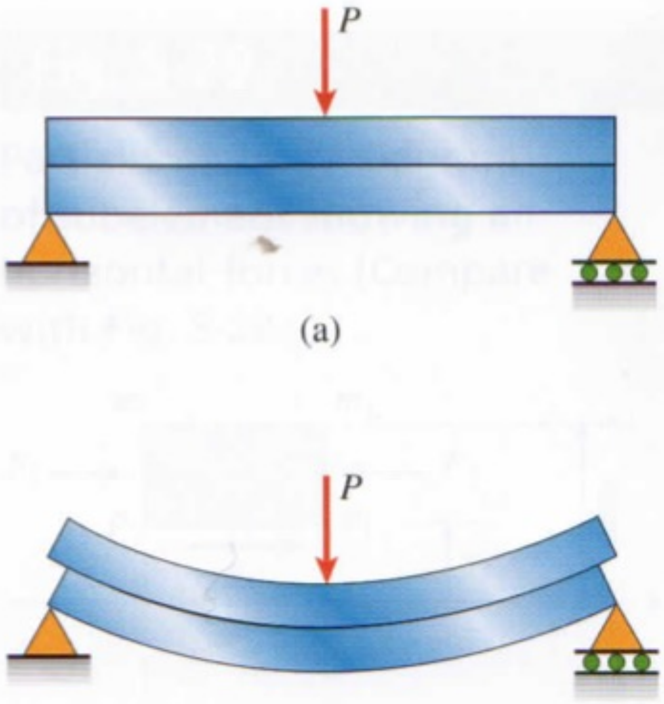
Lecture 12:
Shear stresses in beams

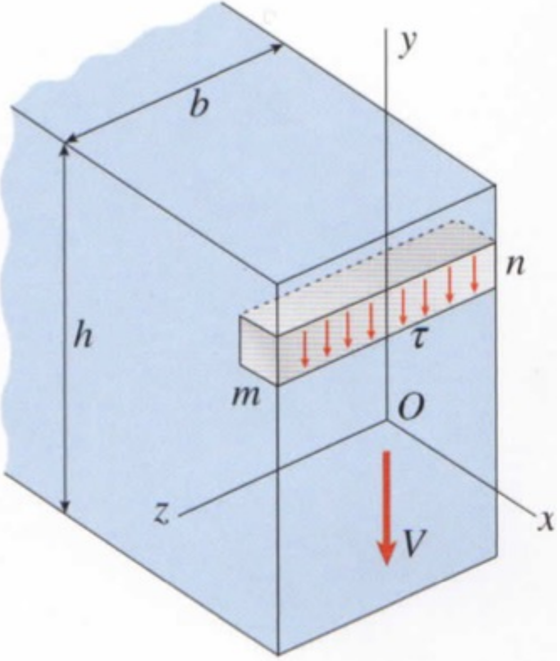


Shear stress in beams

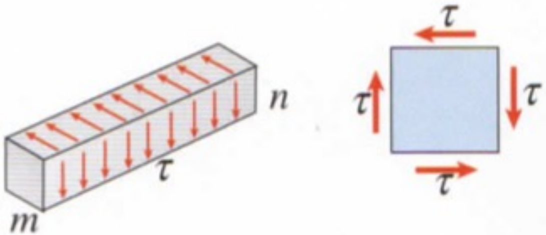
We've seen that if a beam is in pure bending, the only stresses that act on the cross section are normal stresses

In non-uniform bending, we will have normal stresses and shear stresses

Assume two identical beams in bending. Under load, one beam will slide over the other. The force that is required to prevent this sliding is the longitudinal shear force.



(a)



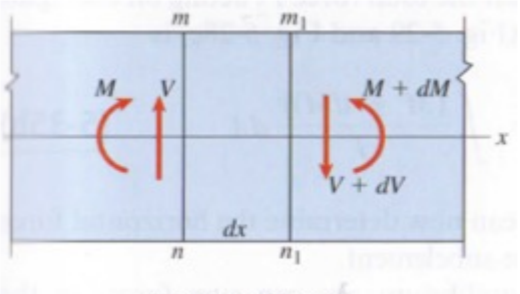
Shear stress in beams

For the case of a rectangular cross-section we can assume:

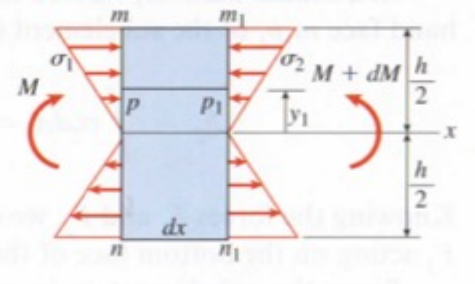
- Shear stresses that act on a cross section are evenly distributed from one side to the other and act vertically
- Shear stresses acting on one side are accompanied by shear stresses of equal magnitude on a perpendicular face
- Therefore we will have vertical AND horizontal shear stresses

First result: at the top and at the bottom of the beam, the horizontal shear stresses are zero. Therefore also the vertical shear stresses are zero.

Shear stress in beams



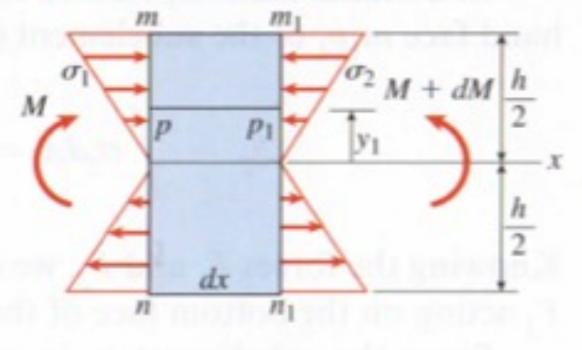
Side view of beam



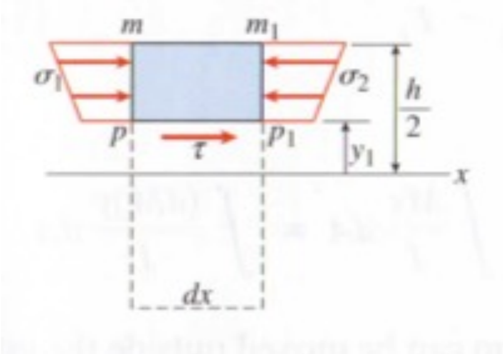
Side view of element

- Assume a beam in non-uniform bending and take two vertical cross-sections a distance *dx* apart.
- Since we have non-uniform bending, we have both normal stresses and shear stresses.
- We want to find an expression that relates external forces to shear stresses
- Our approach:
  - Use the fact that shear stresses act both normal to the beam axis as well as parallel to the beam axis
  - Use the expression for normal stresses (flexure formula) to calculate force-equilibrium in the longitudinal direction in the presence of a moment gradient.

Shear stress in beams



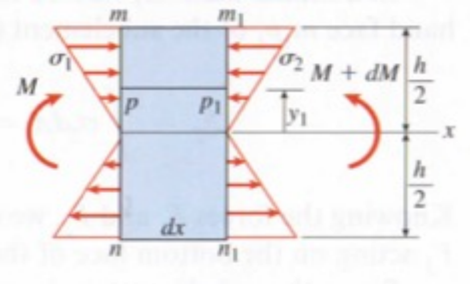
Side view of element



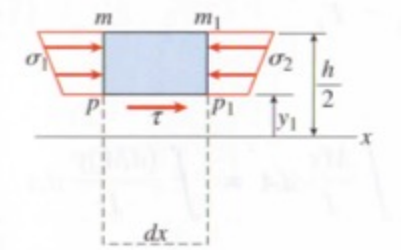
Side view of subelement

- We calculate the normal stresses on each cross-section of the element using the flexure formula
- The stresses vary linearly with y, being zero at y=0
- We know that shear stresses are stresses that occur between two sides of a cross-section. We make a horizontal cut through the element to look at the sub-element mm_1pp_1.
- Calculate the forces acting on the sub-element in the x-direction.

Shear stress in beams



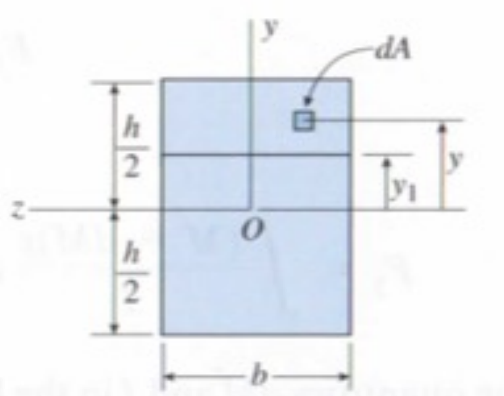
Side view of element



Side view of subelement

- If the bending is uniform, then there is no difference in moment on the face mp and the face m_1p_1. Therefore the normal forces cancel each other out.
- In non-uniform bending, there is a non-zero *dM* between the two cut planes.
- This means that $\sigma_1 \neq \sigma_2$. Which results in a force difference on the two faces.

Shear stress in beams



Cross section of beam at subelement

- We calculate the forces acting on the faces mp and m_1p_1 by integrating over the cross-sectional area of the sub-element.
- Since we are interested in the shear force at a specific value for $y=y_1$, we calculate the integral over the area of $b*y_1$ to $b*h/2$
- Due to the need for equilibrium in the x-direction, the difference between the two forces at the two sides of the sub-element has to be compensated by an additional longitudinal force, which is the shear force between two horizontal slices through the beam.

Shear stress in beams

- From the in-plane shear force we can then calculate the shear stress and find the shear formula:

$$\boxed{\tau = \frac{V \cdot Q}{I \cdot b}}$$

- with Q being the *first moment of area*:

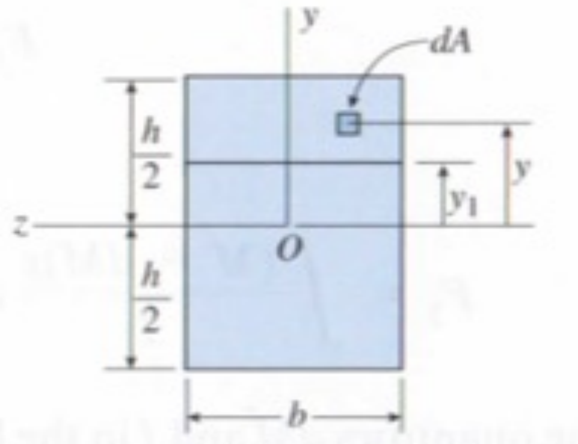
$$\boxed{Q := \int_A y \cdot dA}$$

- The last thing we need to do to describe the shear stress distribution in the beam is to calculate Q.

Shear stress in beams

- For a beam with a rectangular cross-section, we can calculate $Q(y_1)$ by calculating the area from y_1 up to h/2
- We could also calculate the area from y_1 downward, and would bet –Q
- We can easily show that at $y_1=0$, Q is maximum and at $y_1=+/-\ h/2$, Q=0
- Through integration and combination with the shear formula we get:

$$\boxed{\tau(y) = \frac{V}{2I}\left(\frac{h^2}{4} - y^2\right)}$$



Cross section of beam at subelement

Shear stress in beams

$$\boxed{\tau(y) = \frac{V}{2I}\left(\frac{h^2}{4} - y^2\right)}$$

We see now:

- shear stress varies quadratically with the distance from the neutral axis
- shear stress is zero at the beams upper and lower surfaces
- the shear stress is maximum at its neutral axis:

$$\tau_{max} = \frac{Vh^2}{8I} = \frac{3V}{2A}$$

- the maximum shear stress is 50% higher than the average shear stress

Validity of the shear formula

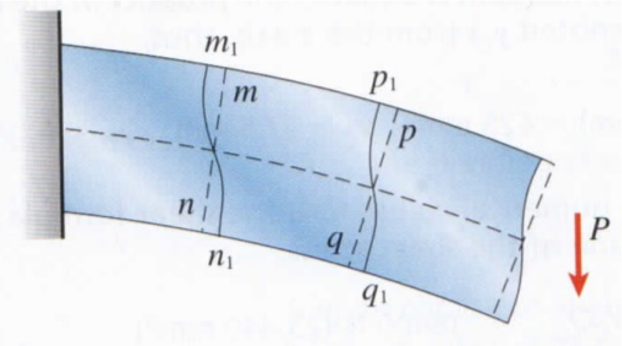
During the derivation of the shear (and flexure) formula we have made a number of assumptions to make the derivation easier.

It is important to use the formula only in cases where these assumptions are justified:

Edges of the cross-section must be parallel to the y axis	The shear stresses must be uniform across the width of the cross-section	The beam must be prismatic (e.g. must have a constant cross-section). The formula is not correct for a tapered beam.

Effect of shear stress

- The shear stress distribution in the beam cross section is quadratic in the plane of bending.
- From Hooke's law in shear we know: $$\gamma = \frac{\tau}{G}$$
- Therefore also the shear strains are not constant along the cross-section. This means that plane cross-sections that were initially normal to the beam axis, will no longer be plane after bending!



Calculating shear and moment diagrams

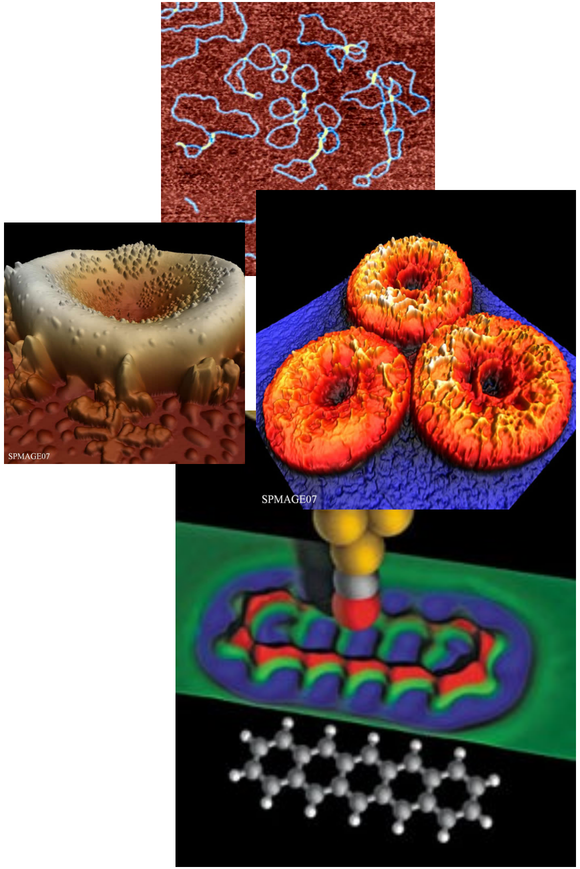
- **Draw** — Draw free body diagram
- **Calculate** — calculate reaction forces
- **Make** — Make a section through the beam and calculate V(x) and M(x) for that section
- **Draw** — Draw the shear and moment diagram below the FBD
- **Check** — Check if your diagram matches with what you know about the support

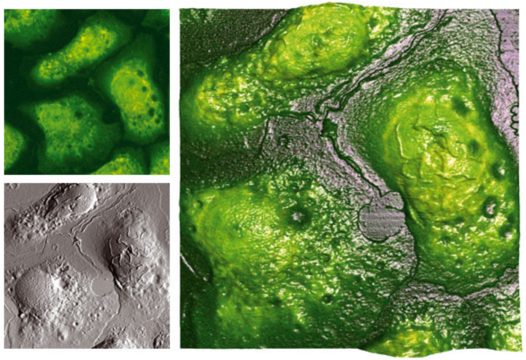
Lecture 12: Beam deflection

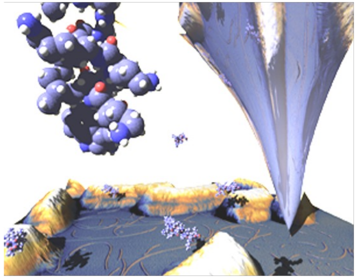
- Applications in nanotechnology
- Governing differential equations
- Solving beam deflection through integration
- Solving beam deflection through superposition
- Statically indeterminate beam deflection

AFM-a versatile tool for nanoscale biology

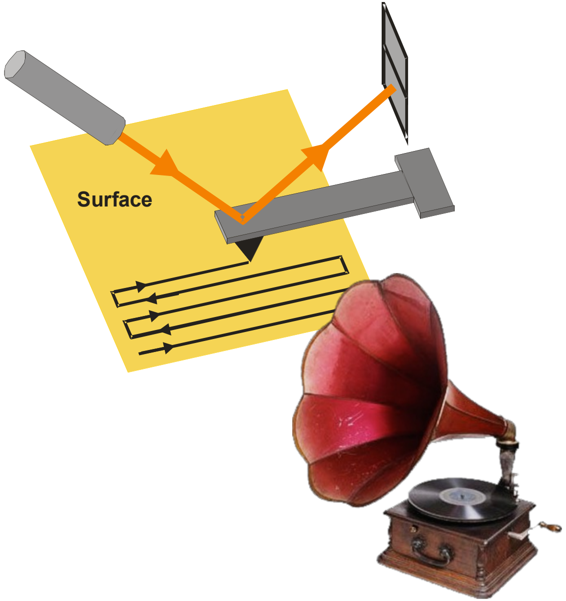
- Single molecule resolution
- High resolution imaging in aqueous solution
- Imaging of living cells
- Single molecule mechanics
- Can be combined with optical microscopy

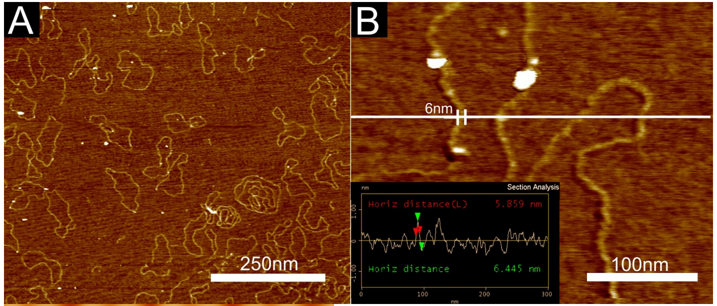


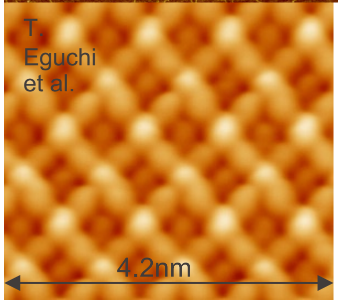




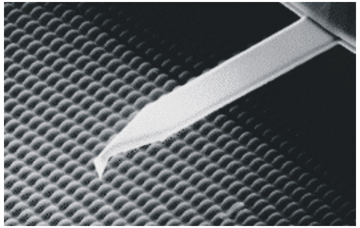
AFM: a Versatile Tool for Nanoscale Measurements





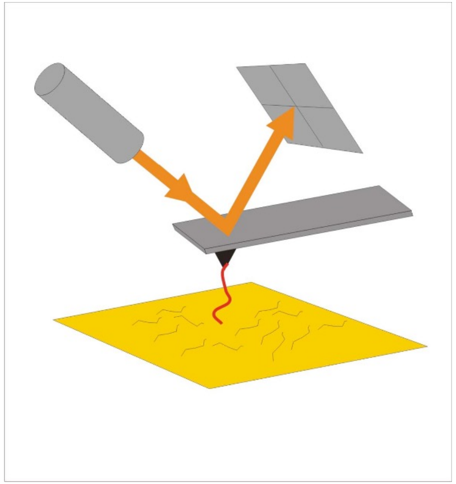


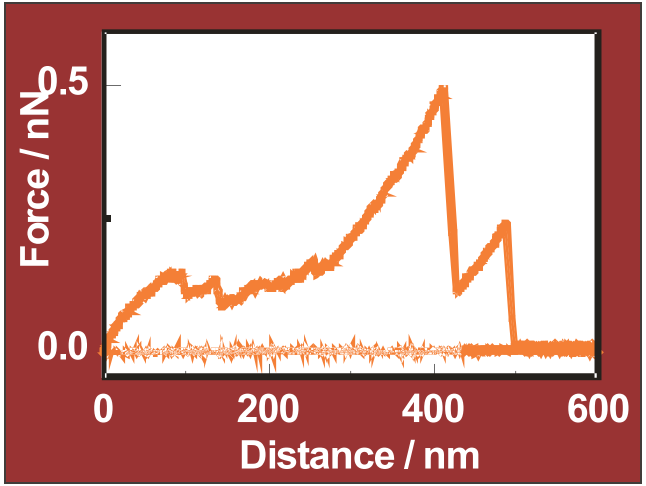
Cantilever



conductivity, surface potential, electrochemical potential, ion currents, magnetism, NMR….and many more

Single Molecule Force Spectroscopy



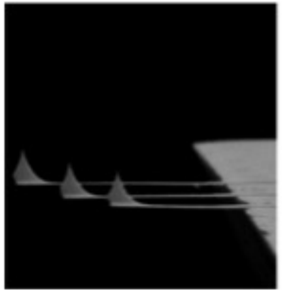


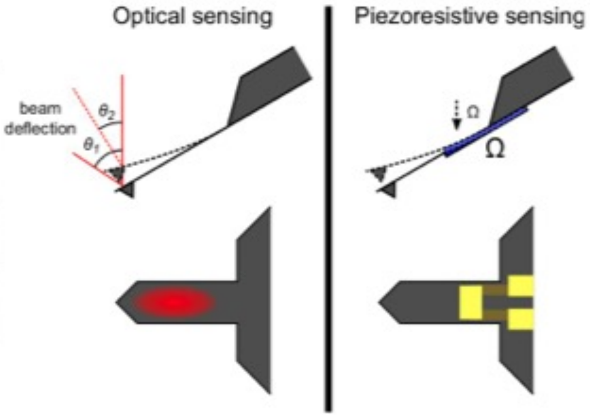
Force resolution: 10s of pN; limited by thermal motion of the cantilever

AFM cantilever beam

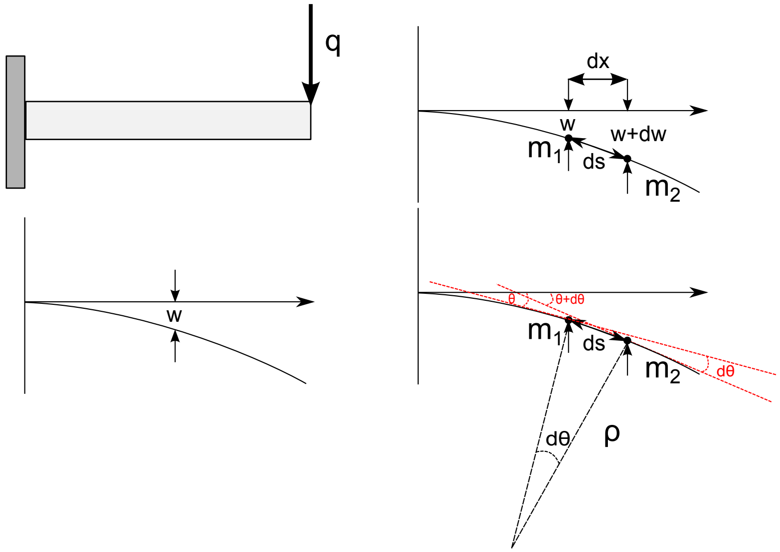
The gate to the nanoworld

- In order to measure very fine features, the cantilever probe needs to be very sharp and sensitive
- The deflection of the cantilever has to be measured very precisely
- Two methods are often used:
  - Optical beam deflection
  - Piezoelectric strain sensing





Beam bending



- We bend the cantilever beam by applying a load at the end
- w(x) describes the amount of deflection of the point on the cantilever from the zero axis
- Two points are a distance ds apart from each-other on the bent beam
- From this we can get a relationship that describes the curvature of the beam

Beam bending - Governing equation

$$\frac{dw}{dx} = -\theta \quad (1)$$

$$\frac{d^2w}{dx^2} = -\frac{M(x)}{EI} \quad (2)$$

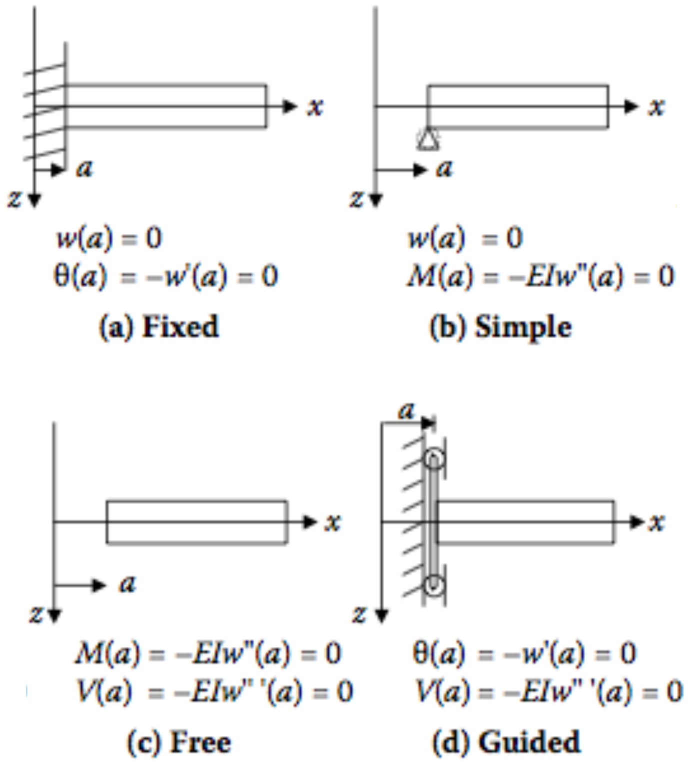
$$\frac{d^3w}{dx^3} = -\frac{V(x)}{EI} \quad (3)$$

$$\frac{d^4w}{dx^4} = \frac{q(x)}{EI} \quad (4)$$

We want to find a relationship between the beam deflection at a point x on the beam as a function of the load

We find 4 differential equations that relate loads to the deflection and the angle

Beam bending-Boundary condition



To solve for the beam bending equation through integration, we need boundary conditions

The type of support of the beam at its end determines the internal forces and moments at the ends, as well as its geometry

We have therefore two types of boundary conditions:

- Static boundary conditions: These come from static equilibrium and pertain to force related quantities (V,M)
- Kinematic boundary conditions: these define the deformational and geometric constraints for the angle and the bending

Beam bending - Abrupt changes

- When we have mathematical discontinuities due to an abrupt change in load or stiffness, we supplement our boundary conditions with the physical requirement that the neutral axis must be continuous!
- Deflection and tangent needs to be the same coming from both sides of the point of discontinuity:

$$\lim_{x \uparrow a} w(a) = \lim_{x \downarrow a} w(a)$$

$$\lim_{x \uparrow a} w'(a) = \lim_{x \downarrow a} w'(a)$$

Beam deflection

- Solving through integration

  - If we want to solve beam equation through integration, we need to integrate 4 times:

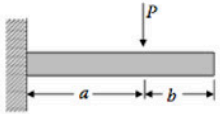
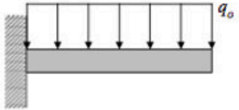
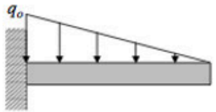
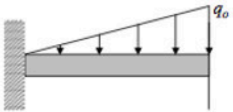
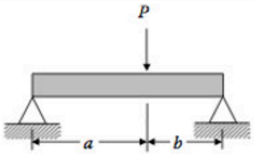
$$\int EIw''''(x)dx = \int q(x)dx$$

$$EIw(x) = \int\int\int\int q(x)dx + \frac{1}{6}C_1x^3 + \frac{1}{2}C_2x^2 + C_3x + C_4$$

  - We know already that:

$$V = -\int q(x)dx + C_1 \qquad M = -\int q(x)dx + C_1x + C_2$$

  - Therefore:
    - We get C1 and C2 from the boundary conditions of M(x) and V(x)
    - We get C3 from the boundary condition of the angle of deflection and C4 from the boundary condition of w

	$w(x)=\frac{PL^3}{6EI}\left[3\left(\frac{x}{L}\right)^2-\left(\frac{x}{L}\right)^3\right]$
	$w(x)=\frac{Pa^3}{6EI}\left[3\left(\frac{x}{a}\right)^2-\left(\frac{x}{a}\right)^3\right]$ $0\le x\le a$ $w(x)=\frac{Pa^3}{6EI}\left[3\left(\frac{x}{a}\right)-1\right]$ $a\le x\le L$
	$w(x)=\frac{q_oL^4}{24EI}\left[6\left(\frac{x}{L}\right)^2-4\left(\frac{x}{L}\right)^3+\left(\frac{x}{L}\right)^4\right]$
	$w(x)=\frac{q_oL^4}{120EI}\left[10\left(\frac{x}{L}\right)^2-10\left(\frac{x}{L}\right)^3+5\left(\frac{x}{L}\right)^4-\left(\frac{x}{L}\right)^5\right]$
	$w(x)=\frac{q_oL^4}{120EI}\left[20\left(\frac{x}{L}\right)^2-10\left(\frac{x}{L}\right)^3+\left(\frac{x}{L}\right)^5\right]$
	$w(x)=\frac{PbL^2}{6EI}\left[\left(1-\left(\frac{b}{L}\right)^2\right)\frac{x}{L}-\left(\frac{x}{L}\right)^3\right]$ $0\le x\le a$

Beam deflection – Solving through superposition

- As long as the beams behave linearly elastic, we are dealing with linear differential equations.
- For such a situation, we can separate a difficult load profile into simpler sub parts:

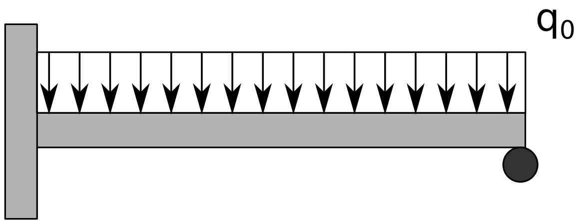
$$q(x)=q_1(x)+q_2(x)+\ldots$$

- We can then do the integrations over the individual q_i separately.
- To find the solution for the deflection due to the complex load profile, we can just sum up the deflections caused by the sub-loads q_i.

$$w(x)=\sum_i w_i(x)$$

- We can tabulate the deflection formulas due to standard loads.

Statically indeterminate beams – Solving through integration

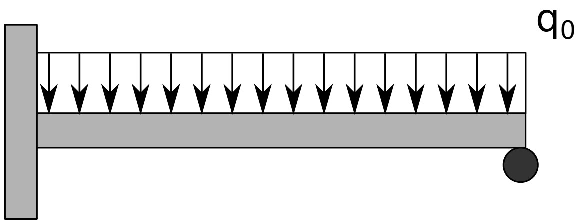


Often beams are supported more that absolutely required for static equilibrium.

A cantilever that is supported also on it's unmounted end is considered a "proper cantilever"

We treat over constrained beams in bending just like normal beams. The static indeterminacy is solved automatically through the use of the boundary conditions to calculate the integration constants.

Example: Statically indeterminate beams



- Solve the following statically indeterminate system through integration of the beam deflection differential equations. Calculate:
  - deflection
  - shear forces
  - bending moments
  - slopes

- Approach:
  - Set up load equation q(x)
  - Integrate the differential equations
  - Solve for the reaction forces using the boundary conditions